Counting in hyperbolic space: number theory and geometry

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Outline

Pell's Equation

- Algorithm for solving $x^2 - 2y^2 = \pm 1$ (Pythagoreans)
- Start with

$$x_1 = 1, \quad y_1 = 1$$

- We get

$$(x_2, y_2) = (3, 2)$$

$$(x_3, y_3) = (7, 5)$$

$$(x_4, y_4) = (17, 12)$$

- Recurrences

$$x_{n+1} = x_n + 2y_n$$

$$y_{n+1} = x_n + y_n$$

- Fundamental solution: $1 + \sqrt{2}$

$$x_n + \sqrt{2} y_n = (1 + \sqrt{2})^n$$

Indefinite Quadratic Forms

- $Q(x, y) = ax^2 + bxy + cy^2, \quad d = b^2 - 4ac > 0$
- Which integers are represented by Q?

$$Q(x, y) = N, \quad x, y, N \in \mathbb{N}$$

Equivalence of Quadratic Forms

$Q \sim Q' \Rightarrow$ they represent the same integers.

- $x^2 - 2y^2 \sim -2x^2 + y^2$
- $x^2 - 2y^2 \sim x^2 + 2xy - y^2$
- $(x + y)^2 - 2y^2 =$
  $x^2 + 2xy - y^2$
  $Q(x + y, y) = Q'(x, y)$
  $Q'(x - y, y) = Q(x, y)$

- $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Class Numbers

$h(d)$ = number of inequivalent forms of discriminant d.

h(d)	d=
1	5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 41, 44, 53
2	40, 60, 65, 85, 104, 105, 120, 136, 140, 156, 165
3	229, 257, 316, 321, 469, 473, 568, 733, 761, 892

- $\mathcal{D} = \{d | d > 0, d \equiv 0, 1 \pmod 4, d \neq \square\}$
- $x^2 - dy^2 = 4$, fundamental solution (t, u)

$$\epsilon_d = \frac{t + u\sqrt{d}}{2}$$

Gauss, Siegel (1944)

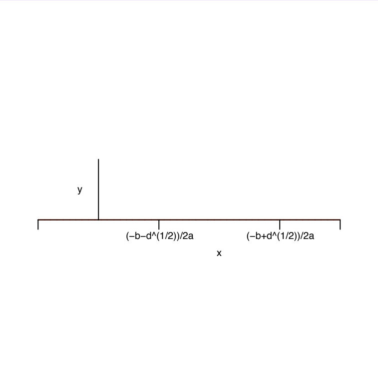
$$\sum_{d \in \mathcal{D}, d \leq x} h(d) \log \epsilon_d = \frac{\pi^2 x^{3/2}}{18\zeta(3)} + O(x \log x).$$

- $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1.$$

From Quadratic Forms to Hyperbolic Geometry

$$Q(x,y) = ax^2 + bxy + cy^2$$

$$az^2 + bz + c = 0 \Leftrightarrow z_1 = \frac{-b+\sqrt{d}}{2a}, \quad z_2 = \frac{-b-\sqrt{d}}{2a}$$



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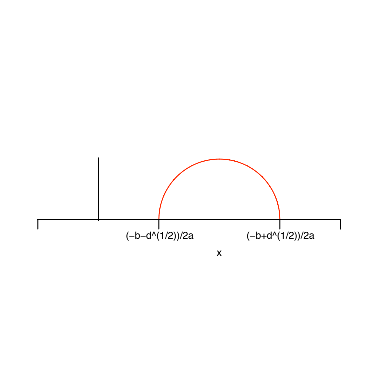


Linear Fractional Transformations and $\mathrm{SL}_2(\mathbb{R})$

$$T(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

maps circles to circles

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to T(z)$$

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1 \right\}$$

acts on

Upper-half space

$$\mathbb{H} = \{z = x + iy, y > 0\}$$

- $z \to z + 1$
- $z \to -\frac{1}{z}$

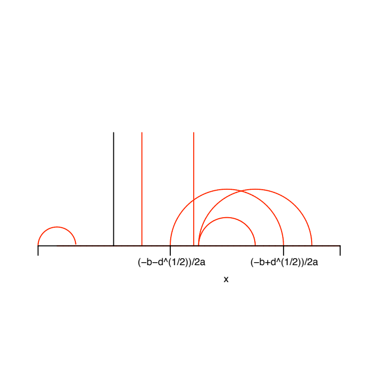
$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \{\gamma \in \mathrm{SL}_2(\mathbb{R}), a, b, c, d \in \mathbb{Z}\}$$

Upper-half space

$$\mathbb{H} = \{z = x + iy, y > 0\} \qquad ds^2 = \frac{dx^2 + dy^2}{y^2}$$

$$z(t) = x(t) + iy(t), \quad t \in [a, b], \quad L = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

Geodesics in the Upper Half Plane



Fundamental Domains

$z \sim w \Leftrightarrow w = T(z), \quad T \in \Gamma$
$\mathbb{H}/\Gamma$ the modular surface

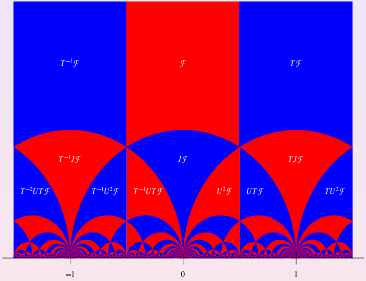


Figure: Fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ and its translates

Equivalence of Quadratic forms and $SL_2(\mathbb{R})$

$$Q(x,y) = \binom{x}{y}^t M \binom{x}{y} \quad M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$$

$$Q' \sim Q \Leftrightarrow M' = \gamma^t M \gamma, \quad \gamma \in \Gamma.$$

$$Q \to g = \begin{pmatrix} \frac{t-bu}{2} & -cu \\ au & \frac{t+bu}{2} \end{pmatrix} \in \Gamma$$

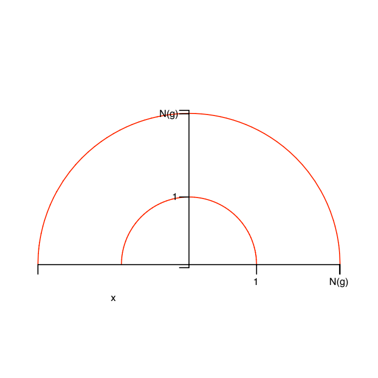
where $t^2 - du^2 = 4$ and (t,u) is the fundamental (smallest solution).

Remarks: 1. g has eigenvalue $\epsilon_d = \dfrac{t + u\sqrt{d}}{2}$

2. Most $g \in SL_2(\mathbb{R})$ can be diagonalised

$$g \sim \begin{pmatrix} N(g)^{1/2} & 0 \\ 0 & N(g)^{-1/2} \end{pmatrix}.$$

Pell's Equation and Lengths of Closed Geodesics



$$\int_1^{N(g)} \frac{1}{y}\, dy = \ln N(g) = \ln(\epsilon_d^2)$$

Theorem

The lengths of the closed geodesics for the hyperbolic surface $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ are $2 \log \epsilon_d$ with multiplicity $h(d)$, $d \in \mathcal{D}$.

Distribution Closed geodesics of $\mathbb{H}/\Gamma$

Closed geodesics γ.

Prime Geodesic Theorem

- $\pi(x) = \{\gamma, \text{length } (\gamma) \leq \ln x\}$
  $\pi(x) \sim \frac{x}{\ln x}, \quad x \to \infty$

Prime Number Theorem

- $\pi(x) = \{p \text{ prime}, p \leq x\}$
  $\pi(x) \sim \frac{x}{\ln x}, \quad x \to \infty$

- Class number distribution (Sarnak, 1982)

$$\sum_{d \in \mathcal{D}, \epsilon_d \leq x} h(d) \sim \frac{x^2}{2 \ln x}$$

The Laplace Operator

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

$$\Delta f = 0 \Leftrightarrow f \quad \text{is harmonic}$$

Eigenvalue problem: Solve

$$\Delta f = \lambda f$$

Infinite Matrix, no determinant to compute eigenvalues.
I require $f(\gamma z) = f(z)$, $\gamma \in \Gamma$ (automorphic form)

Duality between periods and eigenvalues

Duality between periods and eigenvalues

Duality between periods and eigenvalues

Simple trace formula

$$Tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{j=1}^{n} \lambda_j$$

What is and how do I compute the trace of an operator?

Error in PNT and PGT

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log t} dt$$

Prime Number Theorem

$\pi(x) = \{p \text{ prime}, p \leq x\}$

$$\pi(x) = \mathrm{li}(x) + O(xe^{-c\sqrt{\log x}})$$

$$\pi(x) = \mathrm{li}(x) + O(x^{1/2} \log x)$$

$$\Updownarrow$$

RH

Prime Geodesic Theorem

$\pi(x) = \{\gamma, \text{length } (\gamma) \leq \ln x\}$

$\pi(x) - \mathrm{li}(x) =$	$O(x^{3/4})$
Iwaniec	$O(x^{35/48})$
Luo-Sarnak	$O(x^{7/10})$
Sound-Young	$O(x^{25/36})$
Conjecture	$O(x^{1/2+\epsilon})$

Results for $\psi(x)$

$$\Lambda(n) = \begin{cases} \log p, & n = p^k, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi(x) = \sum_{n \le x} \Lambda(n).$$

Cramér (1922)

Assume RH.

$$\frac{1}{A} \int_2^A \left(\frac{\psi(x) - x}{\sqrt{x}} \right)^2 dx = O(1)$$

$$\frac{1}{\log A} \int_2^A \left(\frac{\psi(x) - x}{x} \right)^2 dx \longrightarrow \sum_{\rho} \frac{1}{|\rho|^2}, \quad A \to \infty.$$

Recent results

Define

$$\Lambda_\Gamma(P) = \begin{cases} \log N(P_0), & P = P_0^k, \\ 0, & \text{otherwise.} \end{cases}$$

$$\psi_\Gamma(x) = \sum_{N(P)\leq x} \Lambda_\Gamma(P).$$

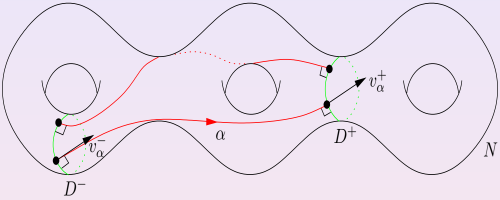
Theorem (Cherubini–Guerreiro 2017)

$$\frac{1}{A}\int_A^{2A} (\psi_\Gamma(x) - x)^2 \, dx = O_\epsilon(A^{5/4+\epsilon})$$

Theorem (Balog–Biró–Harcos–Maga 2018)

$$\frac{1}{A}\int_A^{2A} (\psi_\Gamma(x) - x)^2 \, dx = O_\epsilon(A^{7/6+\epsilon})$$

More counting in hyperbolic space



Work by: A. Good, Parkkonen and Paulin, Martin–Mckee–Wambach, Tsuzuki, Lekkas–Petridis